Set-theoretic Solutions of the Yang-Baxter Equation and Related Quadratic Algebras

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- The Yang-Baxter equation (YBE) is a fundamental equation in Mathematical Physics.
- It was first introduced in the field of statistical mechanics in the late 1960s.

The Yang-Baxter Equation

Let V be a vector space. A linear automorphism $R : V \otimes V \to V \otimes V$ is a solution to the Yang-Baxter equation if the equality $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$, holds in $V \otimes V \otimes V$, where $R_{12} = R \otimes id$, $R_{23} = id \otimes R$.

Set-theoretic solutions to the YBE

- Finding all solutions of the Yang-Baxter equation a difficult task far from its final resolution even for small dimensional vector spaces.
- In 1991 the Fields medalist Drinfeld proposed to study set-theoretical solutions of the Yang-Baxter equation.
- The first influental works devoted to set-theoretical solutions were the papers of Gateva-Ivanova and Van den Bergh in 1998, and Etingof, Schedler, and Soloviev in 1999. These two seminal works gave the start of a very active area of research. Now, every year hundreds of papers are published on this topic.

Preliminaries

Notations

- Let X be a non-empty set and let **k** be a field. We denote by $\langle X\rangle$ the free monoid generated by X, where the unit is the empty word denoted by 1
- $\mathbf{k}\langle X\rangle$ the unital free associative **k**-algebra generated by X.
- For a non-empty set $F \subseteq \mathbf{k}\langle X\rangle$, (F) denotes the two sided ideal of $\mathbf{k}\langle X\rangle$ generated by F.
- When the set X is finite, with $|X| = n$, and ordered, we write $X_n := X = \{x_1, ..., x_n\}$, $\mathbf{k}\langle X_n\rangle$ is the free associative **k**-algebra generated by X_n
- We fix deg-lex order $<$ on $\langle X_n\rangle$, where we set $x_1 < x_2 ... < x_n$
- $\mathbb{N}$ denotes the set of all positive integers
- $\mathbb{N}_0$ denotes the set of all non-negative integers

Preliminaries

Associative Algebra

An **(associative) algebra** A over a field **k**, or an **k-algebra,** is a nonempty set A, together with three operations called **addition** (denoted by $+$), **multiplication** (denoted by juxtaposition) and **scalar multiplication** (also denoted by juxtaposition), for which the following properties hold:

1. A is a vector space over F under addition and scalar multiplication.
2. A is a ring with identity under addition and multiplication.
3. If $r \in F$ and $a, b \in A$, then

$$r(ab) = (ra)b = a(rb)$$

An algebra is **finite-dimentional** if it is finite-dimentional as a vector space. An element $a \in A$ is **invertible** if there is $b \in A$ such that $ab = ba = 1$

Preliminaries

Graded Algebra

An algebra A over **k** is said to be a **graded algebra** if as a vector space over **k**, A can be written in the form:

$$A = \bigoplus_{i=0}^{\infty} A_i$$

for subspaces A_i of A, and where the multiplication is $A_i A_j \subseteq A_{i+j}$, $i, j \in \mathbb{N}_0$

Preliminaries

We shall use the *natural grading by length* on the free associative algebra $\mathbf{k}\langle X_n \rangle$.

$$\mathbf{k}\langle X \rangle = \bigoplus_{m \in \mathbb{N}_0} \mathbf{k}\langle X \rangle_m, \text{ where } \mathbf{k}\langle X \rangle_m = \mathbf{k}X^m.$$

Notations and Definitions

- A polynomial $f \in \mathbf{k}\langle X \rangle$ is **homogeneous of degree m** if $f \in \mathbf{k}X^m$.
- $\mathcal{T} = \mathcal{T}(X_n) := \{x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} | \alpha_i \in \mathbb{N}_0, i \in \{1, ..., n\}\}$ is the set of ordered monomials in $\langle X \rangle$
- $\mathcal{T}_d = \mathcal{T}(X_n)_d := \{x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} \in \mathcal{T} | \Sigma_{i=1}^n \alpha_i = d\}$ stands for the set of ordered monomials of length d.

Groebner basis for ideals in the free associative algebra

Notations and defintions

- Let $f \in \mathbf{k}\langle X \rangle$ is a nonzero polynomial. Its **leading monomial** with respect to deg-lex order $<$ will be denoted **LM**(f). One has $\mathbf{LM}(f) = cu + \Sigma_{1 \le i \le m} c_i u_i$ where $c, c_i \in \mathbf{k}$, $c \neq 0$ and $u > u_i$ in $\langle X_n \rangle$, for every $i \in \{1, ..., m\}$.
- Given a set $F \subseteq \mathbf{k}\langle X \rangle$ of noncommutative polynomials,

$$\mathbf{LM}(F) = \{\mathbf{LM}(f) | f \in F\}$$

.

- A monomial $u \in \langle X \rangle$ is **normal** modulo F if it does not contain any of the monomials **LM**(f) as a subword.
- $\mathcal{N}(F) :=$ the set of all normal words modulo F.

Groebner basis for ideals in the free associative algebra

- Let I be a two-sided graded ideal in $\mathbf{k}\langle X\rangle$ and let $I_m = I \cap \mathbf{k}X^m$. Assume that I is generated by homogeneous polynomials of degree ≥ 2 and $I = \bigoplus_{m\geq 2} I_m$. A monomial $u \in \langle X_n\rangle$ is normal modulo I if it does not contain any of the monomials LM(I) as a subword. We set $\mathcal{N}(I) := \mathcal{N}(\mathbf{LM}(I))$.

Groebner basis for ideals in the free associative algebra

- The free associative algebra $\mathbf{k}\langle X\rangle$ splits as a direct sum of $\mathbf{k}$ vector subspaces:
$$\mathbf{k}\langle X\rangle \simeq \mathrm{Span}_{\mathbf{k}}\mathcal{N}(I) \oplus I$$
and there is isomorphism of vector spaces $A \simeq \mathrm{Span}_k\mathcal{N}(I)$.
- Every $f \in \mathbf{k}\langle X\rangle$ can be written as $f = h + f_0$, where $h \in I$ and $f_0 \in \mathbf{k}\mathcal{N}(I)$. The element f_0 is called the **normal form of f (modulo I)** and is denoted by Nor(f).
- Define $\mathcal{N}(I)_m = \{u \in \mathcal{N}(I) | u \,\text{has length m}\}$. Then,
$$A_m \simeq \mathrm{Span}_k\mathcal{N}(I)_m, \;\; \forall m \in \mathbb{N}_0.$$

Groebner basis for ideals in the free associative algebra

Groebner basis

A subset $G \subseteq I$ of monic polynomials is a **Groebner basis of I**, (with respect to the ordering $<$) if

1. G generates I as a two-sided ideal, and
2. for every $f \in I$ there exists $g \in G$ such that **LM**(g) is a subword of **LM**(f), that is $\mathbf{LM}(f) = a\mathbf{LM}(g)b$, for some $a, b \in \langle X \rangle$.

A Groebner basis G of I is *reduced* if

1. the set $G \setminus \{f\}$ is not a Groebner basis of I, whenever $f \in G$
2. each $f \in G$ is a linear combination of normal monomials modulo $G \setminus \{f\}$.

Groebner basis for ideals in the free associative algebra

1. Every ideal I of $\mathbf{k}\langle X\rangle$ has a unique reduced Groebner basis $G_0 = G_0(I)$ with respect to a $<$. Different orderings can give different Groebner bases on the same ideal.
2. G_0 cane be infinite.

Example: Consider $X = \{x, y\}$ and $I = (yy - xy)$.

- If the ordering is $x > y$, the reduction is $xy \to yy$. The Groebner basis is $G_0 = \{yy - xy\}$.
- If the ordering is $y > x$, the given reduction is $yy \to xy$. Inductively, one can show that the Groebner basis is infinite, including all polynomials of the form

$$y\underbrace{xx\ldots x}_{k \text{ times}}y - \underbrace{xx\ldots x}_{k+1 \text{ times}}y, \forall k \in \mathbb{N}_0.$$

Diamond Lemma and Groebner Basis

Let $G \subseteq \mathbf{k}\langle X\rangle$ be a set of non-commutative polynomials. Let $I = (G)$ and let $A = \mathbf{k}\langle X\rangle/I$. Then the following conditions are equivalent.

1. The set G is a Groebner basis of I.
2. Every element $f \in \mathbf{k}\langle X\rangle$ has a unique normal form modulo G, denoted by Nor G(f).
3. There is an equality $\mathcal{N}(G) = \mathcal{N}(I)$, so there is an isomorphism of vector spaces
$$\mathbf{k}\langle X_n\rangle \simeq I \oplus \mathbf{k}\mathcal{N}(G)$$
4. The image of $\mathcal{N}(G)$ in A is a k-basis of A. In this case A can be identified with the k-vector space $\mathbf{k}\mathcal{N}(G)$, made a k-algebra by the multiplication $a \bullet b := Nor(ab)$.

Quadratic Algebras

Definition

A quadratic algebra is an associative graded algebra $A = \oplus_{i \geq 0} A_i$ over a ground field $\mathbf{k}$ determined by a vector space of generators A_1 and a subspace of homogeneous quadratic relations $R = R(A) \subset V \otimes V$, i.e

$$A = T(V)/(R).$$

We assume that A is finitely generated, so $\dim A_1 < \infty$.
The properties of A will be read off a **presentation** $A = \mathbf{k}\langle X \rangle/(\mathfrak{R})$, where by convention

- X is a fixed finite set of generators of degree 1, $|X| = n$,
- $(\mathfrak{R})$ is the two-sided ideal of relations, generated by a finite set $\mathfrak{R}$ of homogeneous polynomials of degree 2.

PBW Algebras

Definition

A quadratic algebra A is a Poincar'e–Birkhoff–Witt type (PBW) algebra algebra if there exists an enumeration $X = X_n = \{x_1, ..., x_n\}$ of X, such that the quadratic relations $\mathfrak{R}$ form a (noncommutative) Groebner basis with respect to the degree-lexicographic ordering $<$ on $\langle X \rangle$.

In this case the set of normal monomials (mod $\mathfrak{R}$) forms a **k**-basis of A called a **PBW basis** and $x_1, ..., x_n$ (taken exactly with this enumeration) are called **PBW generators** of A.

A special class of PBW algebras that are of great importance for our presentation are **the binomial skew polynomial rings.**

Binomial Skew Polynomial Ring

Definition

A binomial skew polynomial ring is a quadratic algebra $A = \mathbf{k}\langle x_1, ..., x_n \rangle / (\mathfrak{R})$ with precisely $\binom{n}{2}$ defining relations

$$\mathfrak{R}_0 = \{f_{ji} = x_j x_i - c_{ij} x_{i'} x_{j'} | 1 \leq i < j \leq n\}$$

such that
(a) $c_{ij} \in \mathbf{k}^{\times}$.
(b) For every pair $1 \leq i < j \leq n$, the relation $x_j x_i - c_{ij} x_{i'} x_{j'} \in \mathfrak{R}_0$ satisfies $j > i', i' < j'$.
(c) Every ordered monomial $x_i x_j$, occurs as a second term in some relation in $\mathfrak{R}_0$.
(d) $\mathfrak{R}_0$ is the *reduced Groebner basis* of the two-sided ideal $(\mathfrak{R}_0)$, with respect to degree-lexicographic order $<$ on $\langle X \rangle$ or equivalently the overlaps $x_k x_j x_i$, with $k > j > i$ do not give rise to new relations in A.

Quadratic sets and their algebras

Quadratic Sets

Let X be a nonempty set (possibly infinite) and let $r : X \times X \to X \times X$ be a bijective map. By (X, r) we denote **the quadratic set**. The image of (x,y) under r is presented as

$$r(x, y) = ({}^{x}y, x^{y})$$

This formula defines the "left action" $\mathcal{L} : X \times X \to X$, and the "right action" $\mathcal{R} : X \times X \to X$, on X as: $\mathcal{L}_x(y) = ^x y$, $\mathcal{R}_y(x) = x^y$, for all $x, y \in X$.

Quadratic sets and their algebras

(i) (X,r) is **non-degenerate**, if the maps $\mathcal{L}_x$ and $\mathcal{R}_y$ are bijective for each $x \in X$.
(ii)(X, r) is **involutive** if $r^2 = id_X$
(iii)(X,r) is **square-free** if $r(x,x) = (x,x)$ for all $x \in X$
(iv)(X,r) is a set-theoretic solution of the Yang–Baxter equation (YBE) if the braided relation

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$

holds in $X \times X \times X$, where $r^{12} = r \times id_X$, $r^{23} = id_x \times r$. In this case, we refer to (X,r) as a **braided set**.
(v)A braided set (X, r) with r involutive is called a **symmetric set**.
(vi) A non-degenerate symmetric set will be called **a solution**.
(X,r) is the *trivial solution* on X if $r(x,y) = (y,x)$ for all $x, y \in X$.

Let (X, r) be quadratic set. Then r obeys the YBE, that is (X, r) is a braided set if and only if the following three conditions hold for all $x, y, z \in X$:

l1: ${}^{x}({}^{y}z) = {}^{{}^{x}y}({}^{x^{y}}z)$,

r1: $(x^{y})^{z} = (x^{{}^{y}z})^{y^{z}}$,

lr3: $({}^{x}y)^{({}^{x^{y}}z)} = {}^{(x^{{}^{y}z})}(y^{z})$.

The map r is involutive if and only if

inv: ${}^{{}^{x}y}(x^{y}) = x$ and $({}^{x}y)^{x^{y}} = y$.

Quadratic algebras

To each quadratic set (X,r) we associate canonically algebraic objects generated by X and with quadratic relations $\mathfrak{R} = \mathfrak{R}(r)$ naturally determined as
$xy = y'x' \in \mathfrak{R}(r)$ iff $r(x, y) = (y', x')$ and $(x, y) \neq (y', x')$ hold in $X \times X$.

- The monoid $S = S(X, r) = \langle X; \mathfrak{R}(r)\rangle$ with a set of generators X and a set of defining relations $\mathfrak{R}(r)$ is called *the monoid associated with (X,r)*.
- The group $G = G(X, r) = G_X$ *associated with (X,r)* is defined analogously.
- For an arbitrary fixed field **k**, the **k**-algebra associated with (X,r) is defined as $A = A(\mathbf{k}, X, r) = \mathbf{k}\langle X\rangle/(\mathfrak{R}_0)\rangle \simeq \mathbf{k}\langle X; \mathfrak{R}(r)\rangle$. When (X, r) is a solution of YBE the algebra A is also called an Yang-Baxter algebra.

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Key facts

Suppose (X, r) is a nondegenerate, square-free, and involutive quadratic set of order $|X| = n$, and let $A = A(\mathbf{k}, X, r)$ be its quadratic algebra. The following conditions are equivalent:
(1) (X, r) is a solution of the Yang-Baxter equation.
(2) A is a binomial skew polynomial ring, with respect to an enumeration of X.
(3)A is an Artin–Schelter regular PBW algebra, that is
(a) A has polynomial growth of degree n;
(b)A has finite global dimension gl dim A = n
(4)The Hilbert series of A is $H_A(t) = 1/(1-t)^n$
Each of this conditions imply that A is a Noetherian domain.

Theorem

Suppose (X,r) is a nondegenerate symmetric set of order n, and $A = A(\mathbf{k}, X, r)$ *is its Yang-Baxter algebra. Then A is a PBW algebra with a set of PBW generators* $X = \{x_1, x_2, ..., x_n\}$*(enumerated properly) if and only if (X,r) is a square-free solution.*

Thank you for your attention!